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Higher-order modulation effects on solitary wave envelopes in deep water Part 2. Multi-soliton envelopes

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Previous experimental and numerical work indicates that an initially symmetric deep-water wave pulse of uniform frequency and moderately small steepness evolves in an asymmetric manner and eventually separates into distinct wave groups, owing to higher-order modulation effects, not accounted for by the nonlinear Schrödinger equation (NLS). Here perturbation methods are used to provide analytical confirmation of this group splitting on the basis of the more accurate envelope equation of Dysthe (1979). It is demonstrated that an initially symmetric multi-soliton wave envelope, consisting of N bound NLS solitons, ultimately breaks up into N separate groups; a procedure is devised for determining the relative speed changes of the individual groups. The case of a bi-soliton ($N = 2$) is discussed in detail, and the analytical predictions are compared to numerical results.

1. Introduction

It was first pointed out by Feir (1967), and later confirmed by the more extensive experimental study of Su (1982), that an initially symmetric deep-water wavepacket of uniform frequency and moderate steepness evolves in an asymmetric manner and ultimately breaks up into separate groups. This asymmetric development is accelerated as the wave steepness is increased, and cannot be accounted for by the familiar nonlinear Schrödinger equation (NLS): according to the NLS, an initially symmetric wavepacket remains symmetric; moreover, if the packet has uniform frequency initially, the long-time asymptotic form of the envelope consists, in general, of a finite number of solitons that are bound together and undergo multi-period recurrence (Zakharov & Shabat 1972). In the further development of the theory, Lo & Mei (1985), on the basis of detailed comparisons of numerical solutions with experiments, concluded that the observed asymmetric evolution of finite-amplitude wavepackets can be explained theoretically using a more accurate envelope equation, derived earlier by Dysthe (1979), which takes into account higher-order modulation effects. The numerical results of Lo & Mei (1985) also clearly indicate that wave-group separation is caused by a relative shift in the carrier frequency, and hence in the group velocity, of each group, in agreement with the experiments of Feir (1967) and Su (1982).

In a recent paper Akylas (1989, hereinafter referred to as I), following a somewhat different theoretical approach, studied the long-time evolution of a solitary wave group of the NLS by perturbation methods, making use of the fact that the higher-order terms in the Dysthe equation are relatively small. According to the perturbation theory, an NLS soliton eventually transforms to a wave group which has lower peak amplitude and moves faster than the original pulse owing to a downshift in its carrier frequency, in quantitative agreement with numerical results. These findings are consistent with previous work in so far as they suggest that, for general initial disturbances with more than one solitary group present, as in the experiments of Feir (1967) and Su (1982), group splitting is caused by a relative shift in the carrier frequency of each group. On the other hand, it is clear that the theory presented in I is not directly applicable to these experimental observations because it does not take into account interactions between individual wave groups in determining the corresponding frequency shifts.

To remedy this difficulty, the perturbation approach of I is extended in the present work to study higher-order modulation effects on multi-soliton disturbances. In particular, the long-time evolution of an initially symmetric wave envelope of uniform frequency, consisting of N bound NLS solitons, is investigated, and a procedure is devised for calculating the corresponding amplitude changes, frequency shifts, and speed changes. Detailed results are presented for a bi-soliton envelope ($N=2$), which reveal that the interaction between the two groups has an appreciable effect on the frequency shifts.

In the following analysis, it proves most convenient to discuss the long-time behaviour of wave envelopes which initially are multi-soliton bound states because, according to the NLS, these envelopes undergo multi-period recurrence in a reference frame moving with the group velocity. However, it is worth emphasizing that, from the viewpoint of an asymptotic theory, this choice of initial conditions does not amount to loss of generality: solution of the NLS by the inverse scattering method (Zakharov & Shabat 1972) shows that, in general, a localized wavepacket with initially uniform frequency will evolve to a bound soliton state after a long time, when higher-order modulation effects are expected to come into play; thus, using a multi-soliton bound state as initial condition for the Dysthe equation provides the appropriate asymptotic matching condition for describing the long-time evolution of a general localized wavepacket with initially uniform frequency. The validity of the perturbation theory is confirmed in §4, by comparing the analytical predictions for a bi-soliton to numerical results.

2. Formulation

As explained in I, the equation derived by Dysthe (1979) for the envelope $A(\xi,\eta)$ of a two-dimensional wavepacket of small steepness ϵ $(0<\epsilon\ll 1)$, propagating on deep water $(-\infty<x<\infty,\ -\infty<y<0)$, takes the dimensionless form

$$A_\eta+\mathrm{i}A_{\xi\xi}+\mathrm{i}A^2A^*+8\epsilon AA^*A_\xi+2\mathrm{i}\epsilon A\mathscr{H}\{AA^*\}_\xi=0 \tag{1}$$

in a frame of reference moving with the group velocity,

$$\xi=2X-T,\quad \eta=\epsilon X,$$

where $X=\epsilon x, T=\epsilon t$ are the 'slow' space and time variables associated with the evolution of the envelope. Here $\mathscr{H}$ stands for the Hilbert transform

$$\mathscr{H}\{A\}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A(\rho)}{\rho-\xi}\,\mathrm{d}\rho,$$

the integral being interpreted as a principal value, and * denotes the complex conjugate. Setting $\epsilon=0$ in (1), one obtains the NLS.

In the experiments of Feir (1967) and Su (1982), a wavemaker was used to generate symmetric wavepackets of constant frequency, and the subsequent wave evolution was monitored by recording free-surface displacements at certain downstream locations. So, for the theoretical formulation to be consistent with these experiments, the wave envelope is prescribed at $\eta = 0$,

$$A = A_0(\xi) \quad (\eta = 0, -\infty < \xi < \infty), \tag{2}$$

with A_0 real and symmetric about $\xi = 0$, and the evolution of the envelope downstream ($\eta > 0$) is to be determined by solving the Dysthe equation (1) subject to the 'initial' condition (2). Our interest centres on the asymptotic form of the wave envelope far from the wavemaker ($\eta \gg 1$), where higher-order terms in (1) are expected to become important in the limit $\epsilon \rightarrow 0$. For this purpose, A_0 is taken to consist of a finite number, N say, of NLS solitons that are bound together; as already remarked in §1, this choice also ensures matching with the wave disturbance near the wavemaker.

Multi-soliton envelopes are well-known solutions of the NLS (Zakharov & Shabat 1972) having the general form

$$A = \tilde{A}(\xi, \eta)\, \mathrm{e}^{\mathrm{i}\psi_N}, \tag{3}$$

where

$$\tilde{A} = 2\surd 2 \sum_{j=1}^{N} \lambda_j \Phi_j \mathrm{e}^{\mathrm{i}\chi_j}, \tag{4}$$

with

$$\lambda_j = \Lambda_j \mathrm{e}^{-\Theta_j/2}, \quad \Theta_j = \kappa_j(\xi + 2\mu_j \eta), \tag{5a}$$

and

$$\chi_j = \psi_j - \psi_N, \quad \psi_j = \mu_j \xi + (\mu_j^2 - \kappa_j^2)\eta. \tag{5b}$$

The $\Phi_j (j = 1, \ldots, N)$ in (4) are determined by solving the linear algebraic system

$$\sum_{l=1}^{N} M_{jl} \Phi_l + \Phi_j = \lambda_j \quad (j = 1, \ldots, N), \tag{6}$$

where

$$M_{jl} = 4\lambda_j \lambda_l \sum_{n=1}^{N} \frac{\lambda_n^{*2}}{m_{jln}} \exp[\mathrm{i}(\chi_l - \chi_n)], \tag{7a}$$

with

$$m_{jln} = [\mu_j - \mu_n + \mathrm{i}(\kappa_j + \kappa_n)][\mu_n - \mu_l - \mathrm{i}(\kappa_l + \kappa_n)]. \tag{7b}$$

Each N-soliton solution depends on $4N$ real parameters: κ_j are related to the soliton amplitudes a_j through

$$\kappa_j = \frac{a_j}{\surd 2} \quad (j = 1, \ldots, N), \tag{8a}$$

μ_j specify the soliton speeds c_j,

$$c_j = -2\mu_j \quad (j = 1, \ldots, N), \tag{8b}$$

while the magnitudes and phases of the complex constants $\Lambda_j (j = 1, \ldots, N)$ are related, respectively, to shifts in the positions and phases of the solitons. In particular, for a bound soliton state, in view of (8*b*), we set

$$\mu_j = 0 \quad (j = 1, \ldots, N), \tag{9}$$

so that the solitons do not separate and remain stationary; in addition, the parameters Λ_j are taken to be real and such that the envelope is symmetric about $\xi = 0$, in accordance with the conditions imposed on A_0 in (2). With this choice of parameters, a bound N-soliton solution of the NLS depends on N parameters, the κ_j $(j = 1, \ldots, N)$, say, and can be written as

$$A(\xi, \eta) = U_0(\boldsymbol{\theta}, \boldsymbol{\tau}) \exp(-\mathrm{i}\tfrac{1}{2}a_N^2 \eta), \tag{10}$$

where

$$\theta_j = \kappa_j \xi \quad (j = 1, \ldots, N); \quad \tau_j = \Omega_j \eta, \quad \Omega_j = \tfrac{1}{2}(a_N^2 - a_j^2) \quad (j = 1, \ldots, N-1). \tag{11}$$

It follows from (3)–(7) that U_0 is a 2π-periodic function in $\tau_j (j = 1, \ldots, N-1)$. Moreover, in addition to the symmetry

$$U_0(-\boldsymbol{\theta}, \boldsymbol{\tau}) = U_0(\boldsymbol{\theta}, \boldsymbol{\tau}), \tag{12}$$

the real and imaginary parts of U_0 also obey

$$\mathrm{Re}\{U_0(\boldsymbol{\theta}, \boldsymbol{\tau})\} = \mathrm{Re}\{U_0(\boldsymbol{\theta}, -\boldsymbol{\tau})\}, \quad \mathrm{Im}\{U_0(\boldsymbol{\theta}, \boldsymbol{\tau})\} = -\mathrm{Im}\{U_0(\boldsymbol{\theta}, -\boldsymbol{\tau})\}. \tag{13}$$

For $N = 1$, U_0 is steady and, making use of (3)–(9), (10) reduces to the stationary isolated soliton, considered in I:

$$A = a_1 \operatorname{sech} \theta_1 \exp(-\mathrm{i}\tfrac{1}{2}a_1^2 \eta); \tag{14}$$

for $N = 2$, U_0 is 2π-periodic in τ_1 and a bi-soliton envelope is obtained (Peregrine 1983):

$$A = \frac{2}{D}\frac{a_2 - a_1}{a_1 + a_2}(a_1 \mathrm{e}^{\mathrm{i}\tau_1} S_1 + a_2 S_2) \exp(-\mathrm{i}\tfrac{1}{2}a_2^2 \eta), \tag{15a}$$

where
$$D = 1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} + \frac{4a_1 a_2}{(a_1 + a_2)^2}(S_1 S_2 \cos \tau_1 - R_1 R_2) \tag{15b}$$

with the shorthand notation $S_{1,2} \equiv \operatorname{sech} \theta_{1,2}, R_{1,2} \equiv \tanh \theta_{1,2}$.

3. Perturbation theory

The fact that an NLS multi-soliton bound state of the form (10) is periodic in $\boldsymbol{\tau}$ and localized in ξ suggests using a multiple-scales perturbation procedure to describe the evolution of the wave envelope far downstream ($\eta \gg 1$). For $\epsilon \ll 1$, it is anticipated that higher-order envelope modulations will have a small effect over a distance of a few periods, and, thus, it is possible to set up an asymptotic theory, analogous to the one developed by Ablowitz & Benney (1970) for multiply periodic nonlinear wavetrains.

The key step is to treat η, τ_j $(j = 1, \ldots, N-1)$ as independent variables so that

$$\frac{\partial}{\partial \eta} \to \frac{\partial}{\partial \eta} + \mathrm{D}_\tau, \quad \mathrm{D}_\tau \equiv \sum_{j=1}^{N-1} \Omega_j \frac{\partial}{\partial \tau_j},$$

and write
$$A(\xi, \eta; \epsilon) = U(\xi, \eta, \boldsymbol{\tau}; \epsilon) \exp(-\tfrac{1}{2}a_N^2 \eta), \tag{16}$$

where U is 2π-periodic in $\boldsymbol{\tau}$ and localized in ξ. Substituting (16) into (1), it is found that U satisfies

$$U_\eta + \mathrm{D}_\tau U - \tfrac{1}{2}\mathrm{i}a_N^2 U + \mathrm{i}U_{\xi\xi} + \mathrm{i}U^2 U^* + 8\epsilon U U^* U_\xi + 2\mathrm{i}\epsilon U \mathscr{H}\{UU^*\}_\xi = 0, \tag{17}$$

and, in view of (10), the initial condition (2) becomes

$$U = U_0(\xi, \boldsymbol{\tau}; \boldsymbol{\kappa}) \quad (\eta = 0). \tag{18}$$

After the above changes of variables are made, the general strategy for solving (17) subject to (18) perturbatively is similar to that followed in I for a single soliton

($N = 1$), although the details are technically more involved for $N \geqslant 2$ because U_0 is periodic in τ rather than steady. To begin with, U is expanded in a power series in ϵ:

$$U = U_0(\xi, \tau; \kappa) + \epsilon U_1(\xi, \eta, \tau) + \epsilon^2 U_2(\xi, \eta, \tau) + \ldots \tag{19}$$

whose leading-order term is given by (18), the known N-soliton bound state of the NLS. Proceeding to $O(\epsilon)$, U_1 satisfies a linear problem consisting of an inhomogeneous equation subject to the quiescent initial condition $U_1 = 0$ at $\eta = 0$. Now, as in I, the solution of this initial-value problem that is localized in ξ and 2π-periodic in τ is expected to exhibit secular behaviour as $\eta \to \infty$; interpreting these non-uniformities in the expansion (19) appropriately will yield the correct asymptotic form of the wave envelope far downstream.

To be more specific, after taking Laplace transforms in η,

$$U_1 = \frac{1}{2\pi \mathrm{i}} \int \hat{U}_1 \, \mathrm{e}^{s\eta} \, \mathrm{d}s,$$

the $O(\epsilon)$ problem for $\hat{U}_1$, in vector form, reads

$$s\mathbf{Q}\hat{\boldsymbol{U}}_1 + \mathbf{L}\hat{\boldsymbol{U}}_1 = \frac{1}{s}\boldsymbol{Z}, \tag{20}$$

where

$$\boldsymbol{Z} = -8|U_0|^2 \mathbf{Q}\boldsymbol{U}_{0\xi} - 2\mathscr{H}\{|U_0|^2\}_\xi \boldsymbol{U}_0. \tag{21}$$

Here $\mathbf{Q}$, $\mathbf{L}$ are 2×2 matrix operators with elements

$$\mathrm{Q}_{11} = \mathrm{Q}_{22}^* = -\mathrm{i}, \quad \mathrm{Q}_{12} = \mathrm{Q}_{21} = 0,$$

$$\mathrm{L}_{11} = \mathrm{L}_{22}^* = -\mathrm{i}\mathrm{D}_\tau + \frac{\partial^2}{\partial \xi^2} + 2|U_0|^2 - \tfrac{1}{2}a_N^2, \quad \mathrm{L}_{12} = \mathrm{L}_{21}^* = U_0^2,$$

operating on two-dimensional column vector functions with entries given by the function itself and its complex conjugate.

It is important to note that $\mathbf{L}$ is a self-adjoint operator, and any vector functions $\boldsymbol{u}, \boldsymbol{v}$ that are 2π-periodic in τ and localized in ξ satisfy

$$\langle \boldsymbol{u}, \mathbf{L}\boldsymbol{v} \rangle = \langle \boldsymbol{v}, \mathbf{L}\boldsymbol{u} \rangle, \tag{22}$$

where the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is defined as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_0^{2\pi} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi (uv^* + u^*v).$$

Furthermore, if $\boldsymbol{u}$ is also a homogeneous solution of $\mathbf{L}$, then, by the standard orthogonality argument, it follows from (22) that, for the inhomogeneous problem

$$\mathbf{L}\boldsymbol{v} = \boldsymbol{r}$$

to have a non-secular solution (i.e. 2π-periodic in τ and localized in ξ), it is necessary that

$$\langle \boldsymbol{u}, \boldsymbol{r} \rangle = 0. \tag{23}$$

In particular, $2N$ independent, non-secular, homogeneous solutions of $\mathbf{L}, \boldsymbol{u}_j$ ($j = 1, \ldots, 2N$) can be readily found in terms of $\boldsymbol{U}_0(\boldsymbol{\theta}, \tau)$:

$$\boldsymbol{u}_j = \frac{\partial \boldsymbol{U}_0}{\partial \theta_j} \quad (j = 1, \ldots, N); \quad \boldsymbol{u}_j = \frac{\partial \boldsymbol{U}_0}{\partial \tau_{j-N}} \quad (j = N+1, \ldots, 2N-1); \quad \boldsymbol{u}_{2N} = \mathbf{Q}\boldsymbol{U}_0, \tag{24}$$

and, making use of the symmetries of U_0 noted earlier in (12) and (13), it is easy to show that $\boldsymbol{Z}$, defined in (21), is orthogonal to all these homogeneous solutions:

$$\langle \boldsymbol{u}_j, \boldsymbol{Z} \rangle = 0 \quad (j = 1, \ldots, 2N). \tag{25}$$

Moreover, in a similar way, it can be shown that

$$\langle \boldsymbol{u}_j, \mathbf{Q}\boldsymbol{u}_k \rangle = 0 \quad (j, k = 1, \ldots, 2N). \tag{26}$$

Now, returning to the inhomogeneous problem (20), we need to examine the asymptotic behaviour of $\boldsymbol{U}_1$ for large η in order to identify possible secular terms which would cause the expansion (19) to become non-uniform. To this end, as explained in I, $\hat{\boldsymbol{U}}_1$ is expanded in powers of s as follows:

$$\hat{\boldsymbol{U}}_1 = \frac{1}{s^2}\sum_{j=1}^{2N} C_j \boldsymbol{u}_j + \frac{1}{s}\boldsymbol{p}_{-1}(\xi, \tau) + \boldsymbol{p}_0(\xi, \tau) + o(1), \tag{27}$$

where $\boldsymbol{u}_j$ are the well-behaved homogeneous solutions of $\mathbf{L}$ defined in (24), and $C_j (j = 1, \ldots, 2N)$ are as yet undetermined constants. Substituting (27) into (20), it is found that $\boldsymbol{p}_{-1}$ satisfies

$$\mathbf{L}\boldsymbol{p}_{-1} = -\sum_{j=1}^{2N} C_j \mathbf{Q}\boldsymbol{u}_j + \boldsymbol{Z}.$$

Note that, in view of (25) and (26), the right-hand side of this inhomogeneous equation is orthogonal to all $\boldsymbol{u}_j (j = 1, \ldots, 2N)$ and, therefore, according to the solvability condition (23), there exists a non-secular solution,

$$\boldsymbol{p}_{-1} = \boldsymbol{f}(\xi, \tau) + \boldsymbol{g}(\xi, \tau), \tag{28}$$

say, where

$$\mathbf{L}\boldsymbol{f} = -\sum_{j=1}^{2N} C_j \mathbf{Q}\boldsymbol{u}_j, \quad \mathbf{L}\boldsymbol{g} = \boldsymbol{Z}. \tag{29a, b}$$

Equation (29b) does not seem to be amenable to analytical solution in general. On the other hand, it is possible to solve (29a) explicitly by considering the particular solutions $\boldsymbol{w}_j (j = 1, \ldots, 2N)$:

$$\boldsymbol{w}_j = \frac{\partial \boldsymbol{U}_0}{\partial \kappa_j} \quad (j = 1, \ldots, N); \quad \boldsymbol{w}_j = \frac{\partial \tilde{\boldsymbol{A}}}{\partial \mu_{j-N}} + \xi \boldsymbol{u}_j \quad (j = N+1, \ldots, 2N-1);$$

$$w_{2N} = \frac{\partial \tilde{\boldsymbol{A}}}{\partial \mu_N} - \xi \sum_{n=1}^{N} \boldsymbol{u}_{n+N}, \tag{30}$$

where $\tilde{\boldsymbol{A}}$ has already been defined in (4)–(7) and the derivatives with respect to the parameters μ_j are evaluated at $\mu_j = 0 \quad (j = 1, \ldots, N)$. It can be readily verified that the $\boldsymbol{w}_j$ satisfy the inhomogeneous problems

$$\mathbf{L}\boldsymbol{w}_j = 2\kappa_j \mathbf{Q}\boldsymbol{u}_{j+N} \quad (j = 1, \ldots, N-1), \tag{31a}$$

$$\mathbf{L}\boldsymbol{w}_N = -2\kappa_N\left(\mathbf{Q}\boldsymbol{u}_{2N} + \sum_{l=1}^{N-1} \mathbf{Q}\boldsymbol{u}_{l+N}\right), \tag{31b}$$

$$\mathbf{L}\boldsymbol{w}_j = -2\kappa_{j-N} \mathbf{Q}\boldsymbol{u}_{j-N} \quad (j = N+1, \ldots, 2N). \tag{31c}$$

So $\boldsymbol{f}$ is posed as a linear combination of the $\boldsymbol{w}_j$,

$$\boldsymbol{f}=\sum_{j=1}^{2N} B_j\boldsymbol{w}_j, \tag{32}$$

and, upon substitution into (29a), using (31), the constants B_j are related to C_j:

$$B_j=\frac{1}{2\kappa_j}(C_{2N}-C_{j+N}) \quad (j=1,\dots,N-1); \quad B_N=\frac{C_{2N}}{2\kappa_N}; \quad B_j=\frac{C_{j-N}}{2\kappa_{j-N}} \quad (j=N+1,\dots,2N). \tag{33}$$

Proceeding to $O(1)$ in the expansion (27), taking into account (28) and (32), it is found that $\boldsymbol{p}_0$ satisfies

$$\mathbf{L}\boldsymbol{p}_0=-\mathbf{Q}\boldsymbol{g}-\sum_{j=1}^{2N} B_j\mathbf{Q}\boldsymbol{w}_j,$$

and for this inhomogeneous equation to have a non-secular solution, the orthogonality condition (23) should hold for all $\boldsymbol{u}_j$:

$$\sum_{l=1}^{2N} B_l\langle \boldsymbol{u}_j,\mathbf{Q}\boldsymbol{w}_l\rangle=-\langle \boldsymbol{u}_j,\mathbf{Q}\boldsymbol{g}\rangle \quad (j=1,\dots,2N). \tag{34}$$

This is a linear system of $2N$ algebraic equations for $B_j (j=1,\dots,2N)$, and when these coefficients are found, the unknown constants $C_j (j=1,\dots,2N)$ follow directly from (33).

The algebraic system (34) can be further simplified by making use of the symmetry property (12): according to (24), the first N homogeneous solutions, $\boldsymbol{u}_j (j=1,\dots,N)$, are odd functions of ξ whereas the rest, $\boldsymbol{u}_j (j=N+1,\dots,2N)$, are even. So, from (31), it is clear that the $2N$ functions $\boldsymbol{w}_j$ can be normalized, by adding appropriate linear combinations of homogeneous solutions, so that $\boldsymbol{w}_j (j=1,\dots,N)$ are even and $\boldsymbol{w}_j (j=N+1,\dots,2N)$ are odd in ξ. As a result of these symmetries, the equation set (34) reduces to two uncoupled systems of N equations in N unknowns:

$$\sum_{l=1}^{N} B_{l+N}\langle \boldsymbol{u}_j,\mathbf{Q}\boldsymbol{w}_{l+N}\rangle=-\langle \boldsymbol{u}_j,\mathbf{Q}\boldsymbol{g}^{\mathrm{o}}\rangle \quad (j=1,\dots,N), \tag{35a}$$

$$\sum_{l=1}^{N} B_{l}\langle \boldsymbol{u}_j,\mathbf{Q}\boldsymbol{w}_{l}\rangle=-\langle \boldsymbol{u}_j,\mathbf{Q}\boldsymbol{g}^{\mathrm{e}}\rangle \quad (j=N+1,\dots,2N), \tag{35b}$$

where, in view of (21) and (29b),

$$\mathbf{L}\boldsymbol{g}^{\mathrm{e}}=-2\mathscr{H}\{|U_0|^2\}_\xi U_0, \quad \mathbf{L}\boldsymbol{g}^{\mathrm{o}}=-8|U_0|^2\,\mathbf{Q}U_{0\xi}, \tag{36a, b}$$

$\boldsymbol{g}^{\mathrm{e}}, \boldsymbol{g}^{\mathrm{o}}$ being even and odd in ξ, respectively.

Having determined the constants $C_j (j=1,\dots,2N)$ in (27), the leading behaviour of $\hat{U}_1$ as $s\to 0$ is now known. Following then the same procedure as in I, inverting this Laplace transform and combining (19), (24), (28), and (32), the asymptotic form, correct to $O(\epsilon)$, of U for large η is obtained:

$$U\sim U_0+\epsilon\eta\left[\sum_{j=1}^{N} C_j\frac{\partial U_0}{\partial\theta_j}+\sum_{j=N+1}^{2N-1} C_j\frac{\partial U_0}{\partial\tau_{j-N}}-\mathrm{i}C_{2N}U_0\right]+\epsilon\left[g+\sum_{j=1}^{2N} B_j w_j\right]+\dots \quad (\eta\to\infty). \tag{37}$$

Clearly, the terms proportional to η in (37) become unbounded as $\eta\to\infty$, and, as anticipated, the straightforward expansion (19) of the wave envelope breaks down far from the wavemaker. To interpret these non-uniformities, one may view (19) as an inner expansion, valid for $\eta \ll 1/\epsilon$, and match (37), the outer limit of this inner expansion, to a suitable outer expansion, valid for $\epsilon\eta = O(1)$. This formal approach was discussed in detail in I for a single soliton. However, a simpler procedure is to note that the secular terms in (37) correspond to a Taylor expansion of $U_0(\boldsymbol{\theta}, \boldsymbol{\tau}) \exp(-\mathrm{i}\tfrac{1}{2}a_N^2 \eta)$ for

$$\theta_j \to \theta_j + \epsilon C_j \eta \quad (j = 1, \ldots, N); \quad \tau_j \to \tau_j + \epsilon C_{j+N} \eta \quad (j = 1, \ldots, N-1); \quad a_N \to a_N + \epsilon \frac{C_{2N}}{a_N}.$$

Hence, taking into consideration (5), (37) can be identified as a 'naive' expansion of a wave envelope having N solitary groups, but with parameters slightly shifted from their original values:

$$\mu_j = \epsilon \Delta\mu_j, \quad a_j \to a_j + \epsilon \Delta a_j, \tag{38a}$$

where

$$\Delta\mu_j = \frac{C_j}{2\kappa_j} \quad (j = 1, \ldots, N); \quad \Delta a_j = \frac{C_{2N} - C_{j+N}}{a_j} \quad (j = 1, \ldots, N-1); \quad \Delta a_N = \frac{C_{2N}}{a_N}. \tag{38b}$$

These shifts correspond to frequency and amplitude changes respectively. Also, according to (8b), the frequency shifts result in speed changes,

$$c_j = \epsilon \Delta c_j, \quad \Delta c_j = -2\Delta\mu_j = -\frac{C_j}{\kappa_j} \quad (j = 1, \ldots, N). \tag{39}$$

Therefore, in general, the N groups are expected to separate far from the wavemaker, $\epsilon\eta = O(1)$, in agreement with the previous experimental and numerical work cited in §1. Furthermore, (33) and (35)–(38) imply that, at least to leading order in ϵ, the speed changes (39) depend on envelope modulations only, and are not affected by the wave-induced mean flow.

As discussed in I, for $N = 1$, the speed change of an isolated envelope soliton of the form (14) with peak amplitude a can be calculated explicitly using the above asymptotic theory. The non-secular homogeneous solutions (24) and the particular solutions (30) are given by

$$u_1 = -aRS, \quad u_2 = -\mathrm{i}aS; \quad w_1 = -a\xi RS + \surd 2S, \quad w_2 = \mathrm{i}\xi aS,$$

where $R \equiv \tanh\theta$, $S \equiv \operatorname{sech}\theta$ with $\theta = \kappa\xi$. The solution of (36b) can be found analytically in this case,

$$g^{\mathrm{o}} = \mathrm{i}2^{\frac{3}{2}} a^2 SR,$$

and from (35a) it follows that

$$B_2 = -\frac{\langle \boldsymbol{u}_1, \mathbf{Q}\boldsymbol{g}^{\mathrm{o}} \rangle}{\langle \boldsymbol{u}_1, \mathbf{Q}\boldsymbol{w}_2 \rangle} = -\tfrac{4}{3}a^2;$$

therefore, using (33) and (38), (39), it is concluded that the group speed increases owing to a frequency downshift:

$$\Delta\mu = -\tfrac{4}{3}a^2, \quad \Delta c = \tfrac{8}{3}a^2. \tag{40}$$

On the other hand, for $N \geqslant 2$, one has to resort to numerical solution of the inhomogeneous problem (36b), in order to determine the constants $B_j (j = N+1, \ldots, 2N)$ from (35a), and thereby obtain the speed changes of the N groups; the task becomes tedious as N increases. The case of a bi-soliton envelope ($N = 2$) is discussed below.

4. Bi-soliton envelope

The symmetric bi-soliton envelope (15) includes two solitons with peak amplitudes a_1, a_2 ($a_1 < a_2$, say) that are bound together and undergo recurrence with the period

$$\frac{4\pi}{a_2^2-a_1^2}. \tag{41}$$

In the limit that one of these solitons has relatively small amplitude ($a_1 \ll 1$, $a_2 = O(1)$), there is little interaction between the two groups, the bi-soliton approaches an isolated NLS soliton, and one can use (40) to estimate the speed change of the main group; the speed of the smaller group is expected to remain essentially unchanged. However, if the soliton amplitudes are comparable, the interaction between the two groups is appreciable, and it is of interest to know how the corresponding speed changes are modified. To this end, as already remarked, it is necessary to solve the inhomogeneous problem (36*b*) numerically.

Since the solution of (36*b*), $g^{\mathrm{o}}(\xi,\tau)$ with $\tau = \frac{1}{2}(a_2^2-a_1^2)\,\eta$, is odd in ξ, we need only consider the region $\xi > 0$. Moreover, the real and imaginary parts of g^{o} are taken to be, respectively, odd and even functions of τ; taking into account the fact that g^{o} is 2π-periodic in τ, one then has

$$\mathrm{Re}\,\{g^{\mathrm{o}}(\xi,\tau=0)\} = 0, \quad \mathrm{Re}\,\{g^{\mathrm{o}}(\xi,\tau=\pi)\} = 0, \tag{42a, b}$$

and it suffices to solve in the interval $0 \leqslant \tau \leqslant \pi$. This normalization is consistent with the parity of the right-hand side of (36*b*) and it also specifies $g^{\mathrm{o}}(\xi,\tau)$ uniquely in view of the symmetries of the regular homogeneous solutions (24) following from (12) and (13).

To compute $g^{\mathrm{o}}(\xi,\tau)$, we use a discrete-Green-function technique: after truncating the domain $\xi > 0$ at a suitably large value of $\xi = \xi_\infty$, say, we write

$$g^{\mathrm{o}}(\xi_m,\tau) = g_p^{\mathrm{o}}(\xi_m,\tau) + \sum_{k=1}^{M} G_k\, g_k^{\mathrm{o}}(\xi_m,\tau) \quad (m = 1,\ldots,M), \tag{43}$$

where ξ_m are M equally spaced grid points in $0 < \xi < \xi_\infty$ and G_k ($k = 1,\ldots,M$) are real constants to be determined. Here g_p^{o} is the solution of the problem

$$\mathbf{L}g_p^{\mathrm{o}} = -8|U_0|^2\,\mathbf{Q}U_{0\xi} \quad \text{with} \quad g_p^{\mathrm{o}}(\xi,\tau=0) = 0,$$

while the influence functions g_k^{o} satisfy

$$\mathbf{L}g_k^{\mathrm{o}} = 0 \quad \text{with} \quad g_k^{\mathrm{o}}(\xi_j,\tau=0) = \mathrm{i}\delta_{jk} \quad (j,k = 1,\ldots,M),$$

where δ_{jk} denotes the Kronecker delta. These initial-value problems are readily solved numerically in $0 \leqslant \tau \leqslant \pi$ through a standard marching procedure using a Crank–Nicolson scheme. Note that g^{o}, as posed in (43), satisfies (42*a*) automatically, and, imposing (42*b*), a linear algebraic system for the constants G_k is obtained:

$$\sum_{k=1}^{M} G_k\,\mathrm{Re}\,\{g_k^{\mathrm{o}}(\xi_m,\tau=\pi)\} = -\mathrm{Re}\,\{g_p^{\mathrm{o}}(\xi_m,\tau=\pi)\} \quad (m = 1,\ldots,M).$$

Thus, after solving this system, g^{o} is determined at the grid points from (43).

To find the speed changes of the two groups of a bi-soliton, it remains to solve the 2×2 linear system (35*a*) for the constants B_1, B_2. For this purpose, the homogeneous solutions $u_1,\ldots,u_4$, defined in (24), are readily obtained from (15), and, using (30), the

			Δc_1		Δc_2	
σ	a_1	a_2	Asymptotic	Numerical	Asymptotic	Numerical
1.8	0.333	1.444	1.5	1.6	5.3	5.0
2.0	0.5	1.5	2.7	2.5	5.4	5.0
2.2	0.636	1.545	3.8	3.2	5.3	5.5
2.4	0.75	1.583	4.8	4.0	5.2	5.0

TABLE 1. Speed changes of the groups resulting from initial condition (45) for various values of the parameter σ in the range $\frac{3}{2} < \sigma < \frac{5}{2}$; the numerical results were obtained using $\epsilon = 0.1$

required particular solutions w_3, w_4, consistent with the normalization imposed earlier, are found to be

$$w_3 = -\mathrm{i}\frac{4\sqrt{2}}{D}\frac{a_1 a_2}{(a_1+a_2)^2}(R_1 S_2 + \mathrm{e}^{\mathrm{i}\tau} R_2 S_1) + \xi u_3, \tag{44a}$$

$$w_4 = -w_3 - \xi u_4, \tag{44b}$$

with the same notation as in (15). So, after computing the coefficients of the system (35a) by numerical integration, the constants B_1, B_2 are determined by solving this system, and the desired frequency shifts and speed changes follow them (33), (38), and (39).

The perturbation theory is now used to find the speed changes of the groups resulting from some typical initially symmetric wavepackets. We choose to discuss initial wave envelopes having the form

$$A = \operatorname{sech}\frac{\xi}{\sigma} \quad (\eta = 0), \tag{45}$$

where σ is a parameter that controls the width of the packet. These bell-shaped envelopes are qualitatively similar to those investigated experimentally by Feir (1967). In addition, for the class of initial envelopes (45), the inverse-scattering solution of the NLS can be found analytically (Satsuma & Yajima 1974), so that the soliton amplitudes of the bound state that obtains for $\eta \gg 1$ are known in closed form:

$$a_{N-j+1} = \frac{2}{\sigma}(\sigma - j + \tfrac{1}{2}) \quad (j = 1, \ldots, N), \tag{46}$$

where $N - \frac{1}{2} < \sigma < N + \frac{1}{2}$ for N bound solitons to be present; in particular, for a bi-soliton, $N = 2$ and σ has to lie in the range $\frac{3}{2} < \sigma < \frac{5}{2}$. The speed changes, predicted by the perturbation theory, of the two groups that emerge for various values of σ in this range are listed in table 1, together with estimates of these speed changes obtained from fully numerical solutions of the Dysthe equation (1) for wave steepness $\epsilon = 0.1$. In implementing the perturbation procedure, described above, the solution of (36b), $g^0(\xi, \tau)$, was computed with a resolution of 100 grid points in $0 < \xi < \xi_\infty = 15$ and 50 grid points in $0 < \tau < \pi$. The numerical results were checked by verifying that the numerical solution of the inhomogeneous problem (31c) for $j = 3$, which is of the same form as (36b), agreed with the known analytical solution (44a). On the other hand, the Dysthe equation was solved numerically using a semi-implicit Crank–Nicolson scheme rather than the Fourier method of Lo & Mei (1985); as was also noted previously by Lo (1986), the spectral technique had difficulty

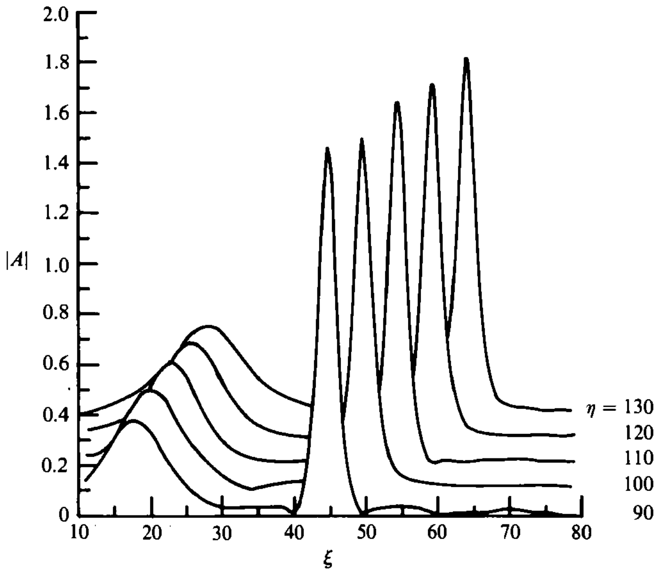

FIGURE 1. Evolution of initial envelope (45) for $\sigma = 2$ as obtained from numerical solution of the Dysthe equation with $\epsilon = 0.1$.

handling the small-amplitude dispersive waves that are radiated by initial conditions of the form (45) ahead and behind the main disturbance. Figure 1 shows the amplitude of the wave envelope, computed for $\sigma = 2$ and $\epsilon = 0.1$, at several locations downstream from the wavemaker where the group splitting has become apparent.

The results in table 1 indicate that the speed changes of both groups are positive and, consequently, their carrier frequencies are downshifted. The group with the larger amplitude is faster, but as σ is increased and, according to (46), the difference in the amplitudes of the two groups becomes smaller, the corresponding speed changes approach each other so that it takes longer for the groups to separate. This is consistent with the experimental observations of Feir (1967) and Su (1982), who noted that the group with the highest amplitude clearly separated from the rest of the disturbance. Also, it is noteworthy that the interaction between the two groups plays an important part in determining the appropriate group-speed changes, especially that of the lower-amplitude group: using (40), which was derived for an isolated soliton, to estimate the speed changes of the two bound solitons with amplitudes given by (46) overpredicts the speed change of the larger group by about 20%; on the other hand, (40) grossly underpredicts the speed change of the lower-amplitude group.

The agreement between the predictions of the perturbation theory for the group-speed changes and the numerical results, listed in table 1, is quite reasonable, given that the value of $\epsilon = 0.1$ is only moderately small – this was a typical value of wave steepness in the experiments and, for this reason, was also used in solving the Dysthe equation numerically. In comparing the asymptotic with the numerical results, it should be kept in mind that the perturbation theory implicitly assumes that the recurrence period (41) of the bi-soliton is small compared with the distance over which higher-order effects become important, $\eta = O(1/\epsilon)$. (In particular, the theory is expected to break down as the two soliton peak amplitudes approach each other because, according to (41), the recurrence period tends to infinity in this limit and the bi-soliton becomes aperiodic.) Now, for the values of σ given in table 1, the corresponding recurrence periods, obtained from (41) and (46), depend on σ very little and are approximately equal to 6.4, which is not all that small compared with $1/\epsilon = 10$ for $\epsilon = 0.1$; nevertheless, the perturbation theory is still useful, as the results in table 1 indicate. Finally, we recall that, in accordance with asymptotic matching, the initial envelope was taken to be a pure bi-soliton in the perturbation theory while, on the other hand, the initial condition (45) corresponds to a bi-soliton, without a dispersive tail, only for $\sigma = 2$ (Satsuma & Yajima 1974); on the basis of the numerical results reported here, neglecting the dispersive tail in the asymptotic theory seems well justified.

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